Lecture 1: Root Systems

§1 Roots

We must start on the level of Lie algebras, rather than Lie groups.

Definition

- A complex, finite dimensional Lie algebra \( g \) is semi-simple if \( g = [g, g] = \{[X, Y] \mid X, Y \in g \} \).

Example \( \Delta_n(\mathbb{C}) = \{X \in \operatorname{Mat}_n(\mathbb{C}) \mid \operatorname{Tr} X = 0 \} \) is semi-simple \((\{X, Y\} = XY - YX)\).

Def: A Cartan subalgebra of a semi-simple Lie algebra \( g \) is a subalgebra \( h \) such that:

- \( h \) is nilpotent, which means \( [h, [h, ..., [h, h] ...]] \) (n brackets) is zero for some \( n \).
- \( h = N_g(h) := \{X \in g \mid [X, h] \in h\} \) (i.e., \( h \) is self-normalizing).

It's usually hard to use this definition, so here are some facts. I'll give some examples at everything in a moment.

Facts

- Cartan subalgebras are abelian \((i.e., [h, h] = 0)\).
- All Cartans are conjugate by an automorphism of \( g \), hence they all have the same dimension.

Def: \( r(k) = \dim(\text{any Cartan}) \), the rank of \( g \).

Notation: Write \( \text{ad}(X)(Y) = [X, Y] \).

Then \( \text{ad}(X) : g \to g \) is a linear map, and so is \( \text{ad}(g) : \text{End}(g) \).

In fact, \( \text{ad}(X, Y) = \text{ad}(X) \circ \text{ad}(Y) - \text{ad}(Y) \circ \text{ad}(X) \), so \( \text{ad} \) is a representation \( g \to \text{End}(g) \).

Fact: \( \text{ad}(h) \) is simultaneously diagonalizable, meaning there's a basis \( (X_i) \) of \( g \) such that:

- \( \text{ad}(h) : h \to \text{End}(h) \)
- For some \( \alpha_i \), \( h \cdot h \cdot X_i = \alpha_i X_i \)

Note that the map \( \alpha_i : h \to h \cdot X_i \) is a linear map \( h \to g \), i.e., a character of \( h \) (because \( [h_i, h_i X_i] = \alpha_i[h_i, X_i] + [h_i, h_i] \cdot X_i = \alpha_i h_i + [1, h_i] X_i \)).

Definition: The nonzero \( \alpha_i \)'s are called the roots of \( g \).

Example \( (\mathfrak{sl}_2) \)

Cartan \( \mathfrak{h} = \langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rangle \). (This is the Lie algebra of the standard maximal torus in \( SL_2(\mathbb{C}) \).) Note \( \dim \mathfrak{h} = 2 \).

The basis of \( X \)'s is: The six elementary matrices \( E_{ij} \) for \( i \neq j \): \( E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \), \( E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \), ..., along with any basis for \( \mathfrak{h} \).

We check:

\[
\text{ad}(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) (E_{12}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = (1-(-1)) E_{12}.
\]

So if we define \( \alpha_{ij} := \) the root for \( \mathfrak{e}_{ij} \), then \( \alpha_{ij}(h_i) = -\alpha_i \).

Similarly, \( \alpha_i(h_j) = -\alpha_i \), for \( i \neq j \), are the other roots.

Example \( (\mathfrak{a}_2) \)

\( \mathfrak{h} = \langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rangle \). The simultaneous eigenvectors consists of \( E_{11}, E_{22}, \ldots, \ldots, E_{ii} \), along with any basis for \( \mathfrak{h} \).

The roots are \( \alpha_{ij} = \text{simultaneous eigenvalue of } h \text{ on } E_{ij} \), given by \( \alpha_{ij}(h_i) = \alpha_i - \alpha_j \).

Fact: A subalgebra \( \mathfrak{h} \) of a semi-simple Lie algebra \( g \) is a Cartan \( \iff \) it is maximal abelian, and \( \text{ad}(\mathfrak{h}) \) is simultaneously diagonalizable. (Maximal abelian) means it is contained in no larger abelian subalgebra, not that it has maximal dimension among abelian subalgebras.)
Properties of roots

Fix a Cartan \( h \) in a semisimple Lie algebra \( g \).

Let \( I \) be the (finite) set of roots of \( h \). Note \( I \subseteq \mathfrak{h}^* = \text{Hom}_\mathbb{C}(h, \mathbb{C}) \).

1. For \( \alpha \in I \), let \( \mathfrak{g}_\alpha = \{ x \in \mathfrak{g} \mid [x, h] = \alpha(x) h \} \) be the \( \alpha \)-eigenspace. Then \( \dim \mathfrak{g}_\alpha = 1 \).

   (In the \( \mathfrak{sl}_3 \) example, \( \mathfrak{g}_\alpha = \mathbb{C} \cdot E_{ij} \)).

2. (Root space decomposition) \( \mathfrak{g} = \bigoplus_{\alpha \in I} \mathfrak{g}_\alpha \).

3. If \( \alpha \) is a root, then so is \(-\alpha\).

   (In the \( \mathfrak{sl}_3 \) example, \( -\alpha_i = \alpha_i \)).

4. \( I \) spans \( \mathfrak{h}^* \).

   (In the \( \mathfrak{sl}_3 \) example, \( \{ \alpha_i \mid i = 1, \ldots, n \} \) is a spanning set.)

Something more general than (3) is true. To explain it, we need to talk about the Killing form, which we needs anyway to justify the facts which have been said already.

Def The Killing form is the symmetric bilinear form on \( \mathfrak{g} \)

\[ \langle X, Y \rangle = \text{Tr} (\text{ad}(X) \circ \text{ad}(Y)) \]

Note \( \text{ad}(X) \circ \text{ad}(Y) \) is a linear endomorphism of the vector space \( \mathfrak{g} \), so we can just take the trace as usual. The Killing form is symmetric because \( \text{Tr}(AB) = \text{Tr}(BA) \).

Thm \( (\mathfrak{g}, \mathfrak{h}) \) is semisimple \( \Leftrightarrow \) \( \langle \cdot, \cdot \rangle \) is nondegenerate.

Fact If \( \mathfrak{g} \) is semisimple, then \( \langle \cdot, \cdot \rangle \) explicit to a perfect pairing on any Cartan \( \mathfrak{h} \).

Therefore we get a canonical isomorphism \( \mathfrak{h} \cong \mathfrak{h}^* \) by \( \mathfrak{h} \rightarrow \langle \cdot, \cdot \rangle \), and hence also a perfect pairing \( \langle \cdot, \cdot \rangle : \mathfrak{h} \times \mathfrak{h} \rightarrow \mathbb{C} \), given explicitly by

\[ \langle (H_{ij}), (H_{kl}) \rangle = \langle H_{ij}, H_{kl} \rangle. \]

(5) If \( \alpha, \beta \) are roots, so is the reflection of \( \beta \) across the hyperplane perpendicular to \( \alpha \), i.e.,

\[ \alpha, \beta \in \mathfrak{h} \Rightarrow \alpha^{\perp} = \beta - \frac{2 \langle \alpha, \beta \rangle}{(\alpha, \alpha)} \alpha. \]

Example Let's compute \( S_{\lambda \mu} (\alpha_{13}) \) for \( \mathfrak{sl}_3 \):

\[ S_{\lambda \mu} (\alpha_{13}) = \alpha_{13} - 2 \frac{\langle \alpha_{13}, \alpha_{13} \rangle}{(\alpha_{13}, \alpha_{13})} \alpha_{13}. \]

To compute the Killing form, we need to make explicit the identification \( \mathfrak{h} \cong \mathfrak{h}^* \) by \( \mathfrak{h} \rightarrow \langle \cdot, \cdot \rangle \), and so we need to compute \( \langle \cdot, \cdot \rangle \) on a basis of \( \mathfrak{h} \).

Let \( H_{13} = (1, -1), \ H_{11} = (0, 1) \). A basis of \( \mathfrak{h} \) consists of \( H_{11}, H_{13}, E_{ij} \) for it. One computes easily:

\[ \text{ad}(H_{11})(E_{ij}) = \begin{cases} 2E_{i+j} & \text{if } (i, j) = (1, 1) \\ 2E_{i+j} & \text{if } (i, j) = (1, -1) \\ 2E_{i-j} & \text{if } (i, j) = (-1, -1) \end{cases}, \]

\[ \text{ad}(H_{13})(E_{ij}) = \begin{cases} -2E_{i} & \text{if } (i, j) = (1, 1) \\ 2E_{i} & \text{if } (i, j) = (1, -1) \\ 2E_{i} & \text{if } (i, j) = (-1, -1) \end{cases}, \]

\[ \text{ad}(H_{11})(h) = \text{ad}(H_{13})(h) = 0. \]

Therefore,

\[ \langle H_{11}, H_{13} \rangle = \text{Tr} (\text{ad}(H_{11}) \circ \text{ad}(H_{13})) = 2(1)(1) + (1)(1) + (1)(1) + (1)(1) = -2 + 2 - 2 = 0. \]

Similarly,

\[ \langle H_{11}, H_{11} \rangle = 2 + 1 + 1 + 1 = 4 = \langle H_{13}, H_{13} \rangle. \]
Let \( \Psi_{12} = \langle H_{12}, \cdot \rangle \in S^3 \), \( \Psi_{23} = \langle H_{23}, \cdot \rangle \in S^2 \). Then

\[
\Psi_{12}(\alpha_1, \alpha_2, \alpha_3) = \rho \left( \frac{\alpha_1}{\alpha_2}, \frac{\alpha_2}{\alpha_3}, \frac{\alpha_3}{\alpha_1} \right)
\]

\[
= \Psi_{12}(\alpha_1, \alpha_0) + \Psi_{12}(-\alpha_1, \alpha_1, \alpha_2)
\]

\[
= \alpha_1 \Psi_{12}(H_{12}) + (\alpha_1 + \alpha_2) \Psi_{12}(H_{23})
\]

\[
= \alpha_1 \langle H_{12}, H_{11} \rangle + (\alpha_1 + \alpha_2) \langle H_{12}, H_{13} \rangle
\]

\[
= 12 \alpha_1 - 6(\alpha_1 + \alpha_2)
\]

\[
= 6(\alpha_1 - \alpha_2)
\]

\[
= 6 \alpha_{12}(\alpha_1, \alpha_2, \alpha_3)
\]

So, \( \Psi_{12} = 6 \alpha_{12} \).

Similarly, \( \Psi_{23} = 6 \alpha_{23} \).

Therefore,

\[
\langle \alpha_{12}, \alpha_{13} \rangle = \langle \frac{1}{6} \Psi_{12}, \frac{1}{5} \Psi_{13} \rangle = \frac{1}{15} \langle H_{12}, H_{13} \rangle = \frac{1}{15} \langle -6, -6 \rangle = \frac{-1}{5}
\]

\[
\langle \alpha_{12}, \alpha_{11} \rangle = \langle \frac{1}{6} \Psi_{12}, \frac{1}{5} \Psi_{11} \rangle = \frac{1}{15} \langle H_{12}, H_{11} \rangle = \frac{1}{15} \langle 12, 12 \rangle = \frac{1}{5}
\]

and so

\[
\frac{2 \langle \alpha_{12}, \alpha_{13} \rangle}{\langle \alpha_{12}, \alpha_{11} \rangle} = \frac{2 \langle -1/5 \rangle}{1/5} = -1
\]

Thus we have computed the number we were looking for, and we can substitute back into the definition of \( S_{\alpha_{11}}(\alpha_{13}) \):

\[
S_{\alpha_{11}}(\alpha_{13}) = \alpha_{13} - \frac{2 \langle \alpha_{12}, \alpha_{13} \rangle}{\langle \alpha_{12}, \alpha_{11} \rangle} \alpha_{12} = \alpha_{13} + \alpha_{12}
\]

But this should be a root, i.e., \( S_{\alpha_{11}}(\alpha_{13}) \) should equal \( \alpha_{1i} \) for some \( i \). However, we see that

\[
(\alpha_{13} + \alpha_{12}) \langle \alpha_1, \alpha_2 \rangle = \alpha_1 \cdot \alpha_1 + \alpha_1 \cdot \alpha_2 = \alpha_1 \cdot \alpha_2 - \alpha_1 \cdot \alpha_3 = \alpha_1 (\alpha_2, \alpha_3)
\]

So

\[
S_{\alpha_{12}}(\alpha_{13}) = \alpha_{13}.
\]

We can compute \( S_{\alpha_{12}} \) on any other root as well:

\[
S_{\alpha_{11}}(\alpha_{1i}) = \begin{cases} 
-\alpha_{1i} & i = 1 \\
\alpha_{11} & i = 13 \\
\alpha_{13} & i = 23 \\
-\alpha_{13} & i = 12 \\
-\alpha_{12} & i = 11 \\
\alpha_{11} & i = 32 \\
\alpha_{12} & i = 21 \
\end{cases}
\]

In general, \( S_\alpha \in O(\mathbb{Z}^3, \langle \cdot, \cdot \rangle) \) (the orthogonal group), \( S_\alpha(\Psi) = \Psi - \frac{\langle \alpha, \Psi \rangle}{\langle \alpha, \alpha \rangle} \alpha \), for general \( \Psi \in S^0 \).

Def \( W_\Psi = \text{subgroup of } O(\mathbb{Z}^3, \langle \cdot, \cdot \rangle) \) generated by \( S_\alpha \)'s for \( \alpha \in \mathbb{Z} \). \( W_\Psi \) is called the Weyl group of \( \Psi \).

(6) \( W_\Psi \) is a Finite group.

Note \( \langle \alpha, \alpha \rangle = 2 \), so \( S_\alpha(\alpha) = \alpha - 2 \alpha = -\alpha \). Therefore (5) \Rightarrow (3).
(7) \( \forall \alpha, \beta \in \mathfrak{h}, \frac{2\langle\alpha, \beta\rangle}{\langle\alpha, \alpha\rangle} \in \mathbb{Z} \), and in fact, 
(8) \( \exists \alpha \in \mathfrak{h} \) for any \( n \in \mathbb{Z} \) (because if \( n \geq 1 \), then \( \frac{2\langle\alpha, \alpha\rangle}{\langle\alpha, \alpha\rangle} = 2n \geq 4 \)).

(9) The \( \mathbb{Z} \)-span of \( \mathfrak{h} \) is a lattice in the \( \mathbb{R} \)-span of \( \mathfrak{h} \). (\( \langle, \rangle \) is real-valued and positive definite on the \( \mathbb{R} \)-span of \( \mathfrak{h} \).)

Def. A root system is an \( \mathbb{R} \)-vector space \( V \) together with a positive definite symmetric pairing \( \langle, \rangle \) and a subset \( \Delta \subseteq V \) st.

(a) \( 0 \notin \Delta \)
(b) \( \Delta \) spans \( \mathfrak{h} \)
(c) \( \forall \alpha, \beta \in \Delta, \frac{2\langle\alpha, \beta\rangle}{\langle\alpha, \alpha\rangle} \in \mathbb{Z} \)
(d) \( \forall \alpha, \beta \in \Delta, \beta - \frac{2\langle\alpha, \beta\rangle}{\langle\alpha, \alpha\rangle} \alpha \in \Delta \)

\( \Delta \) is reduced if also
(e) \( \alpha \in \Delta \) \( \Rightarrow \) \( \alpha \notin \Delta \).

Remark. With some effort, finiteness of \( W_\mathfrak{h} \) (defined the same way) and the fact that \( \frac{2\langle\alpha, \beta\rangle}{\langle\alpha, \alpha\rangle} \in \{0, \pm 1, \pm 2, \pm 3, \pm 4\} \), can be shown for general root systems. Furthermore, \( \frac{2\langle\alpha, \beta\rangle}{\langle\alpha, \alpha\rangle} \geq 4 \) \( \Rightarrow \beta = 2\alpha \), so if \( \Delta \) is reduced, then (8) follows for \( \mathfrak{h} \). (9) follows from the theory of simple roots (see the next section).

Summary:

Given a semisimple Lie algebra \( \mathfrak{g} \), a Cartan \( \mathfrak{h} \in \mathfrak{g} \) is a subalgebra which is maximal abelian and such that \( \text{ad}(\mathfrak{h}) \) is simultaneously diagonalizable.

We get a reduced root system \( (V, \Delta, \langle, \rangle) \) by:

- \( \Delta \) is the nonzero simultaneous eigenvalues of \( \text{ad}(\mathfrak{h}) \) on \( \mathfrak{g} \).
- \( V = \mathbb{R} \)-span of \( \mathfrak{h} \) (so \( \mathfrak{h} \) is \( \mathfrak{g} \).
- \( \langle, \rangle \) is induced from the Killing form: \( \langle [X, Y], Z \rangle = \text{tr} \left( \text{ad}(X) \text{ad}(Y) \right) \) for \( X, Y \in \mathfrak{h} \) (the operator \( \text{ad}(X) \text{ad}(Y) \) acting on \( \mathfrak{h} \)) and then \( \langle X, Y \rangle = \langle [X, Y], 1 \rangle \).
- Then \( \forall \alpha \in \Delta, \langle \alpha, \alpha \rangle = 2 \alpha_{\text{eigen}} \) (or root space) is 1-dimensional.

§3. Positive and simple roots.

Fix a root system \( (V, \Delta, \langle, \rangle) \).

Fix any total order "\( > \)" on \( V \) such that:

- \( \psi_1, \psi_2 > 0 \) \( \Rightarrow \) \( \psi_1 + \psi_2 > 0 \)
- Exactly one of the following holds for any \( \psi \in V \):
  - \( \psi > 0 \), \( -\psi > 0 \), \( \psi = 0 \).

Say \( \psi \) is positive if \( \psi > 0 \).

For instance, fixing a basis \( \psi_1, \ldots, \psi_r \) of \( V \), we can take \( \sum \alpha_i \psi_i > 0 \) if \( \alpha_i = b_i, \ldots, \alpha_1 = b_i, \alpha_i > b_i \), some \( i \).

Let \( \Delta^+ = \{ \alpha \in \Delta \mid \alpha > 0 \} \) - the positive roots. Then \( \Delta^- = \{-\alpha \mid \alpha \in \Delta^+ \} \) has \( \Delta^+ \cup \Delta^- = \Delta \) and \( \Delta^+ \cap \Delta^- = \emptyset \).

Def. A root is simple if it is positive, and cannot be written as a sum of two positive roots.

Let \( \Delta \subseteq \Delta^+ \) be the set of simple roots.
• \( \# \Delta = \dim V \).
• Any \( \beta \in \Delta^+ \) can be (uniquely) written \( \beta = \sum_{\alpha \in \Delta} n_\alpha \alpha \), \( n_\alpha \geq 0 \) \( \forall \alpha \in \Delta \). Hence any \( \gamma \in \bar{\Delta} \) can be written as \( \gamma = \sum_{\alpha \in \Delta} m_\alpha \alpha \)
  where the \( m_\alpha \)'s all have the same sign.
• Since \( 1 \), hence \( \mathbb{R}^+ \), spans \( V \) it follows that \( \Delta \) is \( \mathbb{R} \)-linearly independent, and that the \( m_\alpha \)'s above are unique.

The choice of ordering we started with has an effect on \( \Delta \). But:

Then \( \mathbb{W}_\Delta \) acts simply transitively on the set of all possible \( \Delta \)'s. If \( > \) and \( > \) give the same \( \Delta \), then \( > \) gives the same \( \Delta \). The lexicographical ordering described above with \( \Delta \) as a basis (any order on \( \Delta \)) makes \( \Delta \) simple.

Example (\( \Delta_n \)). - \( \alpha_1, \alpha_2, \ldots, \alpha_{n-1} \) are simple for the lexicographical ordering they induce.
  - Note there are \( n-1 = \text{rk}(\Delta_n) \) of them.

  - Then the positive roots are \( \alpha_{ij}, j > i \).
  - Note \( \alpha_{ij} = \alpha_{i1} + \alpha_{i2} + \cdots + \alpha_{i(n-1)} \) if \( j > i \).
Lecture 2. Root systems (continued).

Recall a (finite dimensional) semisimple Lie algebra $\mathfrak{g}$ and a Cartan $\mathfrak{h} \subseteq \mathfrak{g}$ gives rise to a set of roots

$$\Phi = \{ \text{nonzero simultaneous eigenvalues of } \text{ad}(\mathfrak{h}) \text{ on } \mathfrak{g} \}. $$

For $\alpha \in \mathfrak{h}$, the $\alpha$-eigenspace $\mathfrak{g}_\alpha$ is 1 dim., and

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

Then $(V = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha, \langle \cdot, \cdot \rangle = \text{Killing form})$ is a reduced root system. In particular,

$$s_\alpha(p) = \beta - \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathfrak{h} \quad \forall \alpha, \beta \in \Phi$$

The Weyl group $W = \langle s_\alpha \rangle \leq O(V, \langle \cdot, \cdot \rangle)$ is finite.

A total order $\succ$ on $\Phi$, respecting + and −, determines $\Phi^0 = \{ \alpha \in \Phi \mid \alpha > 0 \}$ and $\Delta = \{ \text{simple roots} \} \subseteq \Phi^0$.

$W$ acts simply transitively on $\{ \Delta \}$, and $\Delta$ determines an order $\prec$.

$\Delta$ is a basis for $V$.

§4. Dynkin Diagrams

Fact: If $\alpha, \beta \in \Delta$, $\alpha \neq \beta$, then $\langle \alpha, \beta \rangle \leq 0$ (hence $\frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \{ 0, -1, -2, -3 \}$ in this case).

Def. The Dynkin diagram is the graph with (vertices) = $\Delta$, and max$\{ \langle \alpha, \beta \rangle, \langle \beta, \alpha \rangle \}$ edges between $\alpha$ and $\beta$, directed towards $\alpha$ if $\langle \alpha, \beta \rangle < 0$.

Example: $(\mathfrak{a}_0)$ Recall $\alpha_i \prec \alpha_j$ if $i < j$.

Therefore the Dynkin diagram is

$$\begin{array}{cccccccccc}
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
\alpha_0 & \overset{1}{\rightarrow} & \alpha_1 & \overset{2}{\rightarrow} & \cdots & \overset{3}{\rightarrow} & \alpha_i & \overset{4}{\rightarrow} & \cdots & \overset{3}{\rightarrow} \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
\end{array}$$

$(\mathfrak{A}_n)$

Def. $\Phi$ is reducible if there are subspaces $V_1, V_2 \subseteq V$ with subsets $\Phi_1 \subseteq \Phi_1$ such that $V = V_1 \oplus V_2$, $\Phi = \Phi_1 \cup \Phi_2$, and $(V_1, \Phi_1, \langle \cdot, \cdot \rangle)$ reduces root systems. Otherwise, $\Phi$ is irreducible.

We say simple Lie algebras $\leftrightarrow$ reduced irreducible root systems $\leftrightarrow$ Connected Dynkin diagrams.

Then: Two reduced, irreducible root systems having the same Dynkin diagram are isomorphic (meaning there's a linear isomorphism between vector spaces preserving the form and sending roots bijectively to roots.)

• The possible connected Dynkin diagrams are the usual ones:

$$\begin{array}{ccccccc}
A_n & \begin{array}{ccc}
\circ & \circ & \cdots & \circ & \circ \\
1 & 2 & \cdots & n-1 & n \\
\end{array} & (n \geq 1) & E_6 & \begin{array}{ccc}
\circ & \circ & \circ & \circ & \circ & \circ \\
1 & 2 & 3 & 4 & 5 & 6 \\
\end{array} & (n \geq 6) & \begin{array}{ccc}
\circ & \circ & \circ & \circ & \circ & \circ \\
1 & 2 & 3 & 4 & 5 & 6 \\
\end{array} \\
B_n & \begin{array}{ccc}
\circ & \circ & \cdots & \circ & \circ \\
1 & 2 & \cdots & n-1 & n \\
\end{array} & (n \geq 2) & E_7 & \begin{array}{ccc}
\circ & \circ & \circ & \circ & \circ & \circ & \circ \\
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\end{array} & (n = 7) & G_2 & \begin{array}{ccc}
\circ & \circ & \circ \\
1 & 2 & 3 \\
\end{array} \\
C_n & \begin{array}{ccc}
\circ & \circ & \cdots & \circ & \circ \\
1 & 2 & \cdots & n-2 & n \\
\end{array} & (n \geq 3) & E_6 & \begin{array}{ccc}
\circ & \circ & \circ & \circ & \circ & \circ & \circ \\
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\end{array} & (n \geq 6) & \begin{array}{ccc}
\circ & \circ & \circ & \circ & \circ & \circ & \circ \\
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\end{array} \\
D_n & \begin{array}{ccc}
\circ & \circ & \cdots & \circ & \circ \\
1 & 2 & \cdots & n-1 & n \\
\end{array} & (n \geq 4) & E_6 & \begin{array}{ccc}
\circ & \circ & \circ & \circ & \circ & \circ & \circ \\
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\end{array} & (n \geq 6) & \begin{array}{ccc}
\circ & \circ & \circ & \circ & \circ & \circ & \circ \\
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\end{array} \\
\end{array}$$

• The Dynkin diagram of $\mathfrak{g}$ determines $\mathfrak{g}$ up to isomorphism.
Recovering from its Dynkin Diagram

We want to find the root system given just the simple roots. We'll use the following tool:

Def Let \( \alpha, \beta \in \Delta \). The root system of \( \alpha \) containing \( \beta \) is

\[
\mathcal{S}_\alpha(\beta) = \{ p + n\alpha \mid n \in \mathbb{Z} \text{ and } \beta + n\alpha \in \mathcal{A} \}
\]

Facts. \( \mathcal{S}_\alpha(\beta) \) has no gaps.

- Let \( p, q \in \mathbb{Z} \) st. \( p + q \in \mathcal{S}_\alpha(\beta) \) \( p \leq n \leq q \). Then

\[
p\cdot q = -\frac{2\langle \alpha, p \rangle}{\langle \alpha, \alpha \rangle}
\]

- If \( \alpha, \beta \in \Delta \), then \( \alpha - \beta \notin \mathcal{A} \), and so \( p = 0 \) and \( q = -\frac{2\langle \alpha, p \rangle}{\langle \alpha, \alpha \rangle} \)

- If \( \gamma = \sum_{\alpha \in \Delta} \epsilon \alpha \in \mathcal{A}^+ \), with \( \sum \epsilon \alpha > 1 \), then we can find \( \beta \in \Delta \) such that \( \gamma - \beta \notin \mathcal{A} \).

These facts will let us recover \( \mathcal{A} \). How about the Lie bracket?

Facts. If \( \alpha, \beta, \alpha + \beta \in \Delta \), then \( [\gamma_\alpha, \gamma_\beta] = \gamma_{\alpha + \beta} \)

\[
[g_\alpha, g_\beta] = C \cdot H_{\gamma} \in \mathfrak{h}^\vee, \quad \text{where } \langle H_{\gamma}, \cdot \rangle = \alpha(\cdot). \quad \text{(In fact } [E_\alpha, \alpha_\alpha] = \langle E_\alpha, \alpha \rangle H_\alpha \text{.)}
\]

Now for \( \alpha \in \Delta \), let:

- \( h_\alpha = \frac{2}{\langle \alpha, \alpha \rangle} H_\alpha \)
- \( e_\alpha \in \mathfrak{g}_\alpha \) be any nonzero vectors
- \( f_\alpha \in \mathfrak{g}_\alpha \) st. \( \langle e_\alpha, f_\alpha \rangle = \frac{2}{\langle \alpha, \alpha \rangle} \).

With these choices, \( C \cdot h_\alpha + C_\alpha + Cf_\alpha \in \mathfrak{h}^\vee \); \( h_\alpha \mapsto (-1), \ e_\alpha \mapsto (0,0), \ f_\alpha \mapsto (0,0) \), and

- \( [h_\alpha, e_\alpha] = \frac{2}{\langle \alpha, \alpha \rangle} \alpha(\alpha) e_\alpha = \frac{2}{\langle \alpha, \alpha \rangle} \langle H_\alpha, H_\alpha \rangle e_\alpha = \frac{2}{\langle \alpha, \alpha \rangle} \langle H_\alpha, \gamma \rangle e_\alpha = \frac{2}{\langle \alpha, \alpha \rangle} \langle \alpha, \alpha \rangle e_\alpha = 2e_\alpha \)
- \( [h_\alpha, f_\alpha] = -2f_\alpha \)
- \( [e_\alpha, f_\alpha] = \langle e_\alpha, f_\alpha \rangle H_\alpha = \frac{2}{\langle \alpha, \alpha \rangle} H_\alpha \in \mathfrak{h} \).

Then (Serre) The vectors \( \{ h_\alpha, e_\alpha, f_\alpha \mid \alpha \in \Delta \} \) generate \( \mathfrak{g} \) as a Lie algebra with precisely the following relations:

1. \( [h_\alpha, h_\beta] = 0 \)
2. \( [e_\alpha, f_\beta] = \delta_{\alpha \beta} h_\alpha \)
3. \( [h_\alpha, e_\beta] = \frac{\langle e_\alpha, f_\beta \rangle}{\langle \alpha, \alpha \rangle} e_\beta \)
4. \( [h_\alpha, f_\beta] = -\frac{\langle e_\alpha, f_\beta \rangle}{\langle \alpha, \alpha \rangle} f_\beta \) (1 plus length of \( \mathcal{S}_\alpha(\beta) \))
5. \( (A[d_\alpha]^{-1})_{-\langle (\alpha, \beta), 2\alpha \rangle} e_\beta = 0 \)
6. \( (A[d_\alpha]^{-1})_{-\langle (\alpha, \beta), 2\alpha \rangle} f_\beta = 0 \)

We give an instructive example on the next page.
Example (g_2). Let \( g_2 \) have Dynkin diagram \( \tilde{A}_3 \). (\( \alpha \) longer).

Find \( \mathfrak{g}^+ \). We have, by the diagram:

\[
\frac{2\langle \alpha, \beta \rangle}{(\alpha, \alpha)} = -1, \quad \frac{2\langle \alpha, B \rangle}{(B, B)} = 3.
\]

So

\[
\mathfrak{g}^+ = \{ \alpha, \beta, \alpha + \beta, \alpha + 2\beta, \alpha + 3\beta \}
\]

Note that since \( \alpha + \beta \not\in \mathfrak{g}^+ \), we have \( \mathfrak{g}^p(\alpha + \beta) = \mathfrak{g}^p(\beta) \). Similarly, \( \mathfrak{g}^p(\alpha) = \mathfrak{g}^p(\alpha + \beta) = \mathfrak{g}^p(\alpha + 2\beta) = \mathfrak{g}^p(\alpha + 3\beta) \).

Then we try taking \( \mathfrak{g}^p \) (new roots) to generate more roots; repeat until we can't find new ones, then all roots are exhausted. We compute:

\[
\mathfrak{g}^p(\alpha + 2\beta) \quad \text{and} \quad \mathfrak{g}^p(\alpha + 3\beta)
\]

This gives us one more root to try:

\[
\mathfrak{g}^p(2\alpha + 3\beta)
\]

We find

\[
\mathfrak{g}^+ = \{ \alpha, \beta, \alpha + \beta, \alpha + 2\beta, \alpha + 3\beta, 2\alpha + 3\beta \}.
\]

So \( \dim \mathfrak{g} = \# \mathfrak{g}^+ = 4 \), and a basis is

\[
h_\alpha, h_\beta, e_\alpha, e_\beta, [e_\alpha, e_\beta], [e_\beta, e_\alpha], [e_\alpha, [e_\alpha, e_\beta]], [e_\beta, [e_\beta, e_\beta]], [e_\alpha, [e_\alpha, [e_\alpha, e_\beta]]], \text{some brackets with } f_\alpha, f_\beta,
\]

Serre relations + Jacobi identity give all possible brackets. For example:

\[
[f_\alpha, [e_\alpha, e_\beta]] = -[e_\beta, [f_\alpha, e_\alpha]] = [e_\alpha, [e_\beta, f_\alpha]]
\]

\[
= -\frac{2\langle \alpha, \beta \rangle}{(\alpha, \alpha)} e_\beta
\]

\[
= e_\beta.
\]
Lecture 3: Reductive Groups.

Fix a field $k \subseteq \mathbb{C}$.

Def. A linear algebraic group over a field $k$ is reductive if its unipotent radical (i.e., the largest closed, connected, normal, unipotent subgroup) is trivial.

Here, linear mean's there's an embedding $G \to \GL_n$ for some $n$, and unipotent means there's an embedding $G \to \langle e_i^\pm 1 \rangle$.

If $G$ is reductive, then

- $\text{Lie } G = \mathfrak{g} = \mathfrak{b} \oplus [\mathfrak{g}, \mathfrak{g}]$
- $G$ is semisimple $\implies G$ is reductive. ("G is semisimple" means "g is semisimple")

§1 Root data

To classify reductive groups, we need slightly more info than for semisimple Lie algs. We'll need four pieces of data:

Def. A torus is an algebraic group $T$ over $k$

$$T_k = G_m, \text{ some } r \geq 0.$$ 

$T$ is split if $T \cong G_m$ (over $k$ instead of $\mathbb{R}$)

Think: $U(1)$ over $\mathbb{R}$ is a circle, not on $\mathbb{R}^x$. But $U(1)^\mathbb{C} = \GL_1(\mathbb{C}) = G_m(\mathbb{C})$. So $U(1)$ is a nonsplit torus over $\mathbb{R}$.

Fix $G$ a connected reductive group over $k$, and fix a maximal torus $T < G$.

Assumption $G$ is split, meaning $T$ is split for some choice of $T$, which we fix.

Here are the four ingredients:

- Characters: Let $X = X^k(T) := \text{Hom}(T, G_m)$ (homomorphisms of alg groups)
- Remark $\mathfrak{t} = \text{Lie } T$ is a Cartan in $\mathfrak{g} = \text{Lie } G$, and $\mathfrak{t} \cap [\mathfrak{g}, \mathfrak{g}]$ is a Cartan in $[\mathfrak{g}, \mathfrak{g}]$. Then if $X: T \to G_m$ is a character, $\text{Lie } X = \mathfrak{t}$ is a character of $\mathfrak{g}$.
- Roots: $\mathbb{G}_a \subseteq \mathfrak{g}_0$, and $T$ is simultaneously diagonalizable on Lie $G$. Let $\Phi := \{\text{roots} \} := \{\text{simple eigenvalues for } T \text{ on } \mathfrak{g}_0 \} \subseteq X$.
- Note: Roots are trivial on $\mathbb{G}_a$.

- Root vectors: $X^*(\mathfrak{t}) := \text{Hom}(G_m, T)$.
- Cocharacters: $X^* = X^k(T)$.
- Cocharacters: $\mathfrak{t} \subseteq \mathfrak{g}_0$, choose root vectors $E_\alpha, E_\alpha^\vee \in [\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}$ for $\alpha$, $-\alpha$.

- Important subgroups $U_\pm, U_x$ with Lie $U_{\pm x} = kE_{\pm x}$

The subgroup $H_x$ generated by $U_\pm, U_x, T$ has derived group $[H_x, H_x] = G_x \cong \text{SL}_2$ or $\text{PSL}_2$.

Then $\exists! x^* : G_x \to G_m$ st $x^* x = (x^*) x$. This is the character associated with $x$.

- Let $\Phi^* = \{ x^* : x \in \Phi \}$. This is the set of cocharacters.

Now define a pairing $\langle x, x^* \rangle : \mathbb{Z} \to \mathbb{Z}$ by $\langle x, x^* \rangle = n$, where $y * x : G_n \to G_m$ is $x \mapsto x^n$. (so $\langle \alpha, \alpha^* \rangle = 2$)

This is a perfect pairing.

$\langle x, x^*, x^* \rangle$ is an example of:
A root datum is a quadruple \((X, P, X^\vee, \Phi)\) with \(X, X^\vee\) free abelian groups, a perfect pairing \((-\cdot): X \times X^\vee \to \mathbb{Z}\), and a bijection \(\Phi \to \Phi^\vee\), denoted by \(x \mapsto x^\vee\), such that:

- \(\forall x \in \Phi, \ (x, x^\vee) = 2\)
- \(\forall x, \beta \in \Phi, \ S_{\alpha}(\beta) = \beta - (\beta, \alpha)^* \in \Phi, \quad \text{and} \quad S_{\beta}(\alpha^\vee) = \alpha^\vee - (\alpha^\vee, \beta) \alpha \in \Phi^\vee\).

Rank \((\alpha, \beta)\) replaces \(\frac{\beta(\alpha)}{\alpha(\beta)}\) from the semisimple Lie algebra story. In particular, \((\alpha, \beta) \neq (\beta, \alpha)\) if \(\alpha\) and \(\beta\) have different lengths.

Let \(\Phi\) need not generate \(X\), or even \(X\mathbb{R}\) over \(R\), due to the center.

**Example (GLn)**: Let \(T = \{e_{ii}, 1 \leq i \leq n\} \subseteq GL_n\). Define \(e_i \in X\) by \(e_i(e_{11}, \ldots, e_{nn}) = e_i\). Then

\[
X = X^\vee(T) = \bigoplus_{i=1}^n e_i \quad \text{(Writing the group law on } X \text{ additively)}
\]

Letting \(\alpha_{ij}(e_{11}, \ldots, e_{nn}) = e_i / e_j\) for \(i \neq j\), then \(\Phi = \{\alpha_{ii}, i \neq j\}\). To find \(U_\alpha\) and \(G_{\alpha}\), let \(E_{ij} \in \text{Lie } GL_n = M_{nn}(k)\) be

\[
E_{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

Then

\[
\text{Ad}(\alpha_{ij})(E_{ij}) = (\alpha_{ij} \cdot e_{ij}) \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \alpha_{ij} \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) E_{ij} = \alpha_{ij}(e_{ij}) E_{ij}
\]

Note: Let \(L = \{e_{ii}\} = k \cdot E_{ii}\), and so \(\langle e_{ii} \rangle\) is \(U_\alpha\). Then

\[G_{\alpha} = \text{group gen by } U_\alpha, \ W_{\alpha}, \text{ and } \langle e_{ii}, e_{ij} \rangle\]

is an SL2. We find

\[
\alpha_{ij}^\vee(x) = \begin{pmatrix} -x & -1 \\ x & 0 \end{pmatrix}
\]

so that \(\alpha_{ij}(\alpha_{ij}^\vee(x)) = 1 / x^{-1} = x^2\).

**Example (SL2)**: Let \(T = \{e_{11}^\vee, e_{22}^\vee\} \subseteq SL_2\).

Let \(e_i \in X\), \(e_i(e_{11}^\vee, e_{22}^\vee) = e_i\). Then

\[
X = X^\vee(T) = \mathbb{Z} e_2.
\]

One checks \(\Phi = \{\pm e_{12}\}\), but \(\alpha_{12}(e_{12}) = e_{11} e_{22} = e_{11}^2 = e_{11}(e_{22})^2\), so alternatively, we can write

\[
\Phi = \{\pm e_{12}, e_{11}^2\} = \{\pm e_{12}, \pm e_{22}\}
\]

The map \(a \mapsto (\alpha_{12})^\vee\) produces \(X^\vee\). But \(x \mapsto (\alpha_{12})^\vee(x) = x^2\), so

\[
\alpha_{12}^\vee(x) = (x, x^2).
\]

Then

\[
X^\vee = \mathbb{Z} \alpha_{12}^\vee
\]

and

\[
\Phi^\vee = \{\pm \alpha_{12}^\vee\}.
\]

Note: \(\mathbb{Z}^\vee = 2 \mathbb{Z} \neq X^\vee\), but \(\mathbb{Z}^\vee \neq X^\vee\).
E_6 (PGL_3), \ T=\{(1, 1)\} = \{(c, 1)\}.

Note \( X \) is gen by \((c, 1)\) \(\mapsto c \). But now

\[ \alpha_{12}(c, 1) = c/1 = c. \]

So

\[ X = \mathbb{Z} \alpha_2, \]

and

\[ \mathfrak{I} = \{ z \alpha_{12} \} \]

\( X' \) is generated by \( e_\nu \), where \( e_\nu(c) = (c, 1) \).

We have \( \alpha_\nu'\nu(\alpha_j) = (\alpha_j, 1) \), because then \( \alpha_\nu'(\alpha_j) = \alpha_j \), so

\[ \mathfrak{I}' = \{ \pm 2 e_\nu \} \]

Note now \( \mathfrak{I} = X \), but \( \mathfrak{I}' = 2X' = X' \).

Remark \( (X', \mathfrak{I}', X, \mathfrak{I}) \) w/ \( (\epsilon, X) = (X, \epsilon) \) is again a root datum, \( \text{the dual root datum.} \)

We see SL_2, PGL_2 are dual. GL_n is self-dual.

Let \( V = \mathbb{R} \otimes X \otimes \mathbb{R} \).

Define \( \langle \eta, \gamma \rangle = \sum \eta \langle x, \gamma \rangle x \).

Then \( \langle \eta, \gamma \rangle = \langle \gamma, \eta \rangle \).

Moreover \( (V, \mathfrak{I}, \langle , \rangle) \) is a root system, and

is the root system of \( [\eta, \gamma] \).

Then Split connected reductive groups over \( k \) are classified up to iso by their root data.

82. Parabolics

Def A Borel subgroup of \( G \) is a maximal, Zariski closed, solvable, connected subgroup of \( G \).

A parabolic subgroup is any subgroup containing a Borel. (Equivalently, \( G/\text{Parabolic} \) is a projective variety, Borel= minimal parabolic.)

To find a Borel \( B \subset G \), choose \( T \), and choose \( > \) on \( \mathfrak{I} \). Then \( T \), along with \( U_\alpha \) for \( \alpha > 0 \), generate \( B \).

Def Let \( P \subset G \) be a parabolic. Define \( N = N_P := R_u(P) := \text{unipotent radical of } P \).

Given a maximal torus \( S \) in \( P \), define also \( M = M_P := Z_P(S) \), called a Levi subgroup of \( P \).

Facts

- \( B \) contains a maximal torus in \( G \), hence \( S \) is a maximal torus in \( G \).

- \( M_P \) depends on \( S \), but any two such \( M_P \)'s are conjugate in \( P \).

- \( P = M_N \).

- \( M \) is reductive.

- \( M_B = S \).

Def Given \( T \) and \( > \) on \( \mathfrak{I} \), the \( P \)'s, containing the Borel \( B \) constructed as above, are called "standard." Then \( S = T \).

E.g. (GL_n) Take \( T = (\cdot \cdot \cdot \cdot) \), and choose \( > \) on \( \mathfrak{I} \) so that \( \alpha_{i,i} \) are the simple roots. Then \( B = (\cdot \cdot \cdot \cdot) \), \( N_B = (\cdot \cdot \cdot \cdot) \), \( M_B = T \).

Standard \( P \)'s are block-upper-triangular matrices:

\[ P = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \]

\[ M_P = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}, \]

\[ N_P = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}. \]
In general, fixing $T \subseteq G$ and $\geq$ on $\Phi$, to get all $P$'s containing $B$, follow this procedure:

Take any subset $\Theta \subseteq \Delta$ of simple roots. Let $P_0$ be generated by $T$ and $U_{\alpha}$ for $\alpha \in \Phi$ satisfying:

If $\alpha = \sum_{\beta \in \Delta} n_{\beta} \beta$, then $n_{\beta} \geq 0$ for all $\beta \in \Theta$. (You're allowed to negate the roots in $\Theta$.)

**Facts:**

- $M_0 = M_{P_0}$ has root system with Dynkin diagram obtained from that of $G$ by removing all nodes not in $\Theta$.
- The $U_{\alpha}$'s in $M_0$ are exactly those coming from $\alpha$'s in $\Phi$ which can be written as $\alpha = \sum_{\beta \in \Delta} n_{\beta} \beta$.
  
  In particular, $U_{\alpha} \subseteq M_0$ for all $\alpha \in \Theta$.

- The $U_{\alpha}$'s in $N_0 = N_{P_0}$ are exactly those coming from $\alpha$'s in $\Phi$ which can be written as $\alpha = \sum_{\beta \in \Delta} n_{\beta} \beta$ with $n_{\beta} \geq 0$ for all $\beta \in \Delta$ and at least one $n_{\beta} > 0$ for some $\beta \in \Theta$. In particular, $U_{\alpha} \subseteq N_0$ for all $\alpha \in \Theta$.
  
  In particular, $U_{\alpha} \subseteq M_0$ and $U_{\alpha} \cdot P_0 = \{1\}$.

$P_\Delta = G$, $P_\Phi = B$.

**Example:**

$T = (\ast \ast \ast)$ usual $P_0 = \text{GL}_4$.

<table>
<thead>
<tr>
<th>$\Theta$</th>
<th>$\phi$</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$\alpha_3$</th>
<th>$\alpha_4$</th>
<th>$\alpha_5$</th>
<th>Color Key</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_0$</td>
<td>$\text{GL}_4$</td>
<td>(XXX)</td>
<td>(XXX)</td>
<td>(XXX)</td>
<td>(XXX)</td>
<td>(XXX)</td>
<td>$M_0$</td>
</tr>
<tr>
<td>$N_0$</td>
<td>$\text{GL}_4$</td>
<td>(XXX)</td>
<td>(XXX)</td>
<td>(XXX)</td>
<td>(XXX)</td>
<td>(XXX)</td>
<td>$N_0$</td>
</tr>
</tbody>
</table>

There are three other standard parabolics, obtained from $\Theta = \{\alpha_3, \alpha_4, \alpha_5\}$, $\{\alpha_1\}$, and $\{\alpha_2\}$.
Lecture 4: Eisenstein Series.

§1 The definition

Let $G$ be a split connected reductive group over $\mathbb{Q}$.
Fix $T \subseteq G$ split maximal torus, let $\Phi(T(G)) \subseteq X(T)$ the root system.
Fix $B \supseteq T$ Borel (equivalently, $\Delta \subseteq \Phi$ simple roots).
Fix $\Theta \subseteq \Delta$, and let $P=P_0$ be the associated parabolic.
Write $P=MN$.

Let $S_0 : M(A) \to R_{>0}$, $S_0(m) = \|\text{det}(m \in \text{Lie}(N(A)))\|$ a modulus character. Extend to $S_0 : P(A) \to R_{>0}$, trivial on $N$.
Let $K \subseteq G(A)$ a maximal compact.

Def For $\tau = (\chi, V_\tau) \in L^2_{\text{cusp}}(M(A),\omega)$ a cuspidal automorphic representation of $M$, let

$$\tilde{T}_\tau^G(\theta, s) = \{ f : G(A) \to V_\tau \mid f \text{ smooth, } K\text{-finite}, f(g) = S_0^{1/2}(g) \text{ for } g \in G(A), \omega(g) \}$$

(again with the same action)

Then define the Eisenstein series by

$$E(\tau, f, s) = \sum_{g \in \text{Par}(G)} f(s)(g) \quad \text{(when convergent)}.$$

Rank $E(\tau, f, s)$ is an automorphic form which transforms under $G(A)$ the same way as $f$.

Fact $E(\tau, f, s)$ converges for $\Re(s) > 0$ and meromorphically continues to $C$.

Example $G = \text{GL}_2$, $P = B = (\pm I)$, $\theta = 1$. Then $\delta_\theta^{(1,1)}(g) = 1/\sqrt{\text{det}(g)}$.

One can choose $f$ s.t., if $g = \left( \begin{array}{cc} 1 & \ast \\ \ast & 1 \end{array} \right)$, then

$$E(g, f, s) = \sum_{g \in \text{Par}(G) \setminus \{1\}} \text{Im}(g \text{det}(g))^{1/2} = \sum_{\text{det}(g) > 1} \frac{g^s}{\sqrt{\text{det}(g)}}.$$
The handouts, Shahidil Assume P, P' are normal. Let \( W(P, P') = \{ w \in W(G, T) | w(\theta) = \theta' \} \). Then

\[
E_p'(g, f, s) = \begin{cases}
    f_s(g) & \text{if } W(P, P') = \{1\} \text{ and } \theta = \theta' \\
    \sum_{w \in W(G, T)} f_s(wng) \, dn' & \text{if } W(P, P') = \{w\}, \text{ but } \theta \neq \theta' \\
    0 & \text{otherwise}
\end{cases}
\]

Hence, \( wP'w^{-1} = P' \).

Idea of proof

\[
E_p'(g, f, s) = \int_{W(G, T) \backslash W(G)} \sum_{w \in W(G, T)} f_s(wng) \, dn'
\]

Use

\[
G(\mathfrak{g}) = \bigcup_{w \in W(G, T)} P(\mathfrak{g}) \sim P(\mathfrak{g})
\]

to split this into a sum over \( w \). Most terms vanish by cospdfinity of \( s, \) what's left is what's above.

§3 Intertwining operators

Recall If \( \sigma \sim \pi' \), then \( C_{\pi'}(\sigma) = \sigma' \circ C_\pi(\sigma) \).

Def For \( w \in W(G, T) \), \( f \in C_\pi(\sigma) \), define the intertwining operators by

\[
M_\pi(\pi', w)(f) = \int_{K_\sigma \times \mathbb{A} \setminus \mathbb{A}} f(w'ng) \, dn' \quad (f \circ f_\pi)
\]

\[
M(\pi, \pi')(f) = \int_{\mathbb{A} \setminus \mathbb{A}} f(w'ng) \, dn'
\]

Then

\[
M_\pi(\pi', w) : C_{\pi'}(\pi) \rightarrow C_{\pi'}(\sigma), \quad M(\pi, \pi') : C_\pi(\sigma) \rightarrow C_\pi(\pi') \quad G\text{-equivariantly, where}
\]

\[
\nu(\pi) = \pi(\nu) = \pi(\nu) \quad \text{and} \quad \nu(\pi') = \pi'(\nu) = \pi'(\nu).
\]

Fact If \( C_{\pi'}(\sigma) \) is unramified and \( f_\pi \in C_\pi(\sigma) \), then \( M_\pi(\sigma', \pi) f_\pi \in C_{\pi'}(\pi) \).

Note: If \( wP'w^{-1} = P' \), then \( w'N^{-1} = N_\pi \sim w'N' w \), so

\[
\int_{wN} f_s(wng) \, dn' = \sum_{w \in wN^{-1} \setminus wN} f_s(wng) \, dn = M(\pi, \nu^{-1}) f_s.
\]

So to compute \( E_p' \), it suffices to compute \( M(\pi, \nu^{-1}) f_s \).

We can do this easily at unramified places.

Recall (Satake) If \( \pi' \) is unramified, then we get \( \chi_{\pi'} : \Gamma(\mathfrak{g}) \rightarrow \mathbb{C}^* \) via the Satake transform.

Thus (Gindikin-Karpelevich, Langlands) Let \( \mathfrak{g} \) be a set of places away from which \( \pi \) and \( C_{\pi}(\mathfrak{g}, s) \) are unramified. Let \( f_\pi \circ f_\mu \circ f_\nu \circ f_\sigma \). Let \( \Sigma \) be the roots in \( \mathfrak{g} \). View \( \mathfrak{k} = X(\mathfrak{t}) \otimes \mathfrak{e} \). Then

\[
M(\pi, \sigma)(f) = \left( \sum_{\mathfrak{m} \in \mathfrak{g}} \left( \prod_{\mathfrak{m} \in \mathfrak{g}} \left( \prod_{\mathfrak{m} \in \mathfrak{g}} (1 - (s_{\mathfrak{m}, \mathfrak{m}})^{-1} \mathfrak{m})^{-1} \right) \right) \right) \circ \otimes_{\mathfrak{m} \in \mathfrak{g}} f_\pi(v) \circ f_\mu(v) \circ f_\nu(v) \circ f_\sigma(v)
\]

Observation: Let \( \mathfrak{m} = \sum_{\mathfrak{m} \in \mathfrak{g}} \left( \prod_{\mathfrak{m} \in \mathfrak{g}} \left( \prod_{\mathfrak{m} \in \mathfrak{g}} (1 - (s_{\mathfrak{m}, \mathfrak{m}})^{-1} \mathfrak{m})^{-1} \right) \right) \circ \otimes_{\mathfrak{m} \in \mathfrak{g}} f_\pi(v) \circ f_\mu(v) \circ f_\nu(v) \circ f_\sigma(v)

\[
\prod_{\mathfrak{m} \in \mathfrak{g}} (1 - (s_{\mathfrak{m}, \mathfrak{m}})^{-1} \mathfrak{m})^{-1} \quad \text{and} \quad \prod_{\mathfrak{m} \in \mathfrak{g}} (1 - (s_{\mathfrak{m}, \mathfrak{m}})^{-1} \mathfrak{m})^{-1}
\]

Then the product above is \( \prod_{\mathfrak{m} \in \mathfrak{g}} (1 - (s_{\mathfrak{m}, \mathfrak{m}})^{-1} \mathfrak{m})^{-1} \).
\[ E_{4}, \text{ Examples} \]

Let \( G = GL_{2, \mathbb{Q}} \), \( P = P(\alpha, \beta, \gamma, \delta) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \). Then \( M = GL_{2, \mathbb{Q}} \). Let \( \pi \) be cuspidal automorphic forms on \( GL_{2} \).

Let \( P' = P \). Then \( W(P, P) = \{(1, w) \mid w \in \mathbb{A}_{\mathbb{Q}}^*; |w| = 1 \} \) where \( w = s_{w} s_{x}, s_{w}, s_{x} \).

Here,

\[ w(\xi) = \alpha_{3}, \quad w(\eta) = \alpha_{1}, \quad w(\omega) = -\omega_{1} - \omega_{3}. \]

Note \( W_{1}(0) = 0, \) \( W_{1}w^{-1} = P^{-1}\begin{pmatrix} \alpha_{3}^{2} & \alpha_{3} \alpha_{4} \\ \alpha_{4} \alpha_{3} & \alpha_{4} \end{pmatrix} \), \( w_{1} = w^{-1} \) (because \( s_{w} \) and \( s_{x} \) commute).

So if \( f = \otimes f_{v} \), \( v \in \mathbb{A}_{\mathbb{Q}}^{*} \), then

\[ E_{P}(g, f, s) = f_{P}(g) \left( \prod_{v \in \mathbb{A}_{\mathbb{Q}}^{*}} \left( \frac{\mathcal{E}(a_{v}, b_{v}, \gamma_{v}, \delta_{v})}{\mathcal{E}(a_{v}, b_{v}, \gamma_{v}, \delta_{v})} \right) \right) \left( \otimes_{v} M(\pi, \omega, w)f_{v} \right) \otimes f_{w}. \]

We now compute the \( L \)-functions, i.e., we compute \( a_{v} \) and \( c_{w} \).

Say, at \( v \)-infinite \( v \),

\[ L_{v}(\pi, s_{v}) = (1 - \beta_{v} P_{v}^{-1} s_{v})^{-1} \left( 1 - \beta_{v} P_{v}^{-1} \right)^{-1} = L_{v}(\pi, s_{v}) = (1 - \beta_{v} P_{v}^{-1} s_{v})^{-1} \left( 1 - \beta_{v} P_{v}^{-1} \right)^{-1}. \]

Then \( \chi_{\pi_{v}}(\beta_{v}) = \beta_{v}, \chi_{\pi_{v}}(\gamma_{v}) = \beta_{v}, \chi_{\pi_{v}}(\delta_{v}) = \beta_{v}. \]

Let \( \xi_{v} = \begin{pmatrix} \alpha_{v} & \beta_{v} \\ \gamma_{v} & \delta_{v} \end{pmatrix} \rightarrow a_{v}. \) Then, as elements of \( \chi X(\pi_{v}) \), \( \chi_{\pi_{v}} = \xi_{v} = 1 + P_{v}^{-1} \xi_{v} \), \( \xi_{v} = \xi_{v}, \chi_{\pi_{v}} = \xi_{v}, \chi_{\pi_{v}} = \xi_{v}. \)

Let \( \Sigma = \{ e_{1}, e_{2}, e_{3}, e_{4}, e_{5}, e_{6}, e_{7}, e_{8} \} \), so \( 2P = \{ e_{1}, e_{2}, e_{3}, e_{4} \}. \) We have

\[ \langle \xi_{v}, e_{1}, e_{2}, e_{3}, e_{4} \rangle = 2 \langle 1, 2, 3, 4 \rangle = 6, \quad \text{and same for other elements of} \Sigma. \]

Now \( \langle \xi_{v}, e_{1}, e_{2}, e_{3}, e_{4} \rangle = \chi_{\pi_{v}}(e_{1}) \chi_{\pi_{v}}(e_{2}) \chi_{\pi_{v}}(e_{3}) \chi_{\pi_{v}}(e_{4}) \chi_{\pi_{v}}(e_{5}) \chi_{\pi_{v}}(e_{6}) \chi_{\pi_{v}}(e_{7}) \chi_{\pi_{v}}(e_{8}) \).

Thus the \( L \)-function quotient in the constant term is a Rankin-Selberg type:

\[ L_{2}(\pi, s_{v}) = \prod_{v \in \mathbb{A}_{\mathbb{Q}}^{*}} \left( 1 - \beta_{v} P_{v}^{-1} s_{v} \right)^{-1} \left( 1 - \beta_{v} P_{v}^{-1} \right)^{-1}. \]

So \( j = 2, a_{v} = 1, \) \( c_{w} = \text{std} \otimes \text{std}. \)

**Concluding Remarks:** (See Ch. 6 in Shahidi, "Eisenstein Series and Automorphic L-Functions")

- There is a formula for the constant term when \( P \) is not maximal. It's a sum over more Weyl elements. It's still in terms of intertwining operators.
- Eisenstein series can be defined for any \( \psi \in X(\pi) \). We'd get \( E_{\pi}(g, f, \nu) \). We just studied the case \( \nu = 5 \).
- \( E_{\pi}(g, f) \) has meromorphic continuation and satisfies a functional equation:

\[ E_{\pi}(g, f, s) = E_{\pi}(g, M(\pi, w)f, -s) \quad (\text{maximality case}) \]

or

\[ E_{\pi}(g, f, \nu) = E_{\pi}(g, M(\pi, w)f, \nu) \quad (\text{general case}). \]

This implies the meromorphic continuation and a functional equation for the constant term.

- One can try to leverage this to study analytic properties of the \( L \)-functions in the constant term. This is the Langlands-Shahidi method.
- There is a formula for \( M_{\pi} \) at \( \nu = 0 \). You get a product of quotients of \( \Omega \)-factors, in terms of \( a_{i}, c_{w} \), and the Langlands parameters of \( \pi \). This is related to Harish-Chandra's \( \zeta \)-function.